

MODELS FOR ELASTIC SHELLS WITH INCOMPATIBLE STRAINS

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ABSTRACT. The three-dimensional shapes of thin laminae in such as leaves, wings, flowers etc. are driven by the differential growth of these objects. The growth takes place through variations in the metric as a function of location in the central plane of the object, and also across the thickness of the sheet. In this paper, we provide a rigorous derivation of asymptotic theories for the shape of residually strained thin lamina with nontrivial curvature, in both the strong and weak curvature regimes, extending hence the discussion in [15]. In particular, we derive the precise conditions under which the model equations proposed for the description of the deployment of petals during the blooming of a flower [21] are valid, and we generalize the classical Föppl-von Kármán energy for prestrained shells with various relative orders of curvature. For weakly curved shells, we show that the derived models depend on the specific choice of scaling laws for the shallowness of the mid-surface and for the growth tensor with respect to the thickness, and lead to different asymptotic theories based on these assumptions. Along the way, we also propose a new matching property for second order infinitesimal isometries on convex weakly shallow shells.

1. INTRODUCTION

Of late, there has been increasing interest in characterizing the growth-induced morphogenesis of low-dimensional structures such as filaments, laminae and their assemblies, which arise routinely in biological systems and their artificial mimics [13, 14, 5, 4]. The physical basis for *morphogenesis* is now classical and elegantly presented in the Thompson’s opus “On growth and form” in terms of a simple principle: differential growth in a body leads to residual strains that will, generically, result in changes in shape of the body. Eventually, the growth patterns will themselves be regulated by the strains, so that this principle be the basis for the physical self-organization of biological tissues. While such questions lay at the interface of biology, chemistry and physics, they are also fundamentally of geometric nature. Indeed, they may be characterized in terms of a variation on a classical theme in differential geometry - that of embedding a shape with a given metric in a space of possibly different dimension [24, 25, 9]. However, the goal now is not only to state the conditions when it might be done (or not), but also to constructively postulate the resulting shapes in terms of an appropriate mechanical theory.

The combination of the separation of scales that arises naturally in slender structures and the constraints associated with the prescription of growth laws that are functions of space (and time), leads to the expectation that the resulting theories ought to be variants of classical elastic plate and shell theories such as the Föppl-von Kármán or the Donnell-Mushtari-Vlasov [1]. That this is the case, has been shown for bodies that are initially flat

1991 *Mathematics Subject Classification.* 74K20, 74B20.

Key words and phrases. non-Euclidean plates, nonlinear elasticity, Gamma convergence, calculus of variations.

and thin i.e. elastic plates with no initial curvature, using analogies to thermoelasticity [22, 20], perturbation analysis [4, 5], and rigorous asymptotic analysis [15]. However, most laminae are naturally curved in their strain-free configurations, which often is a consequence of slow relaxation, perhaps following a previous growth history. Since even infinitesimal deformations of a curved shell will potentially violate isometry relative to its rest state, one expects that differential growth of such an object will likely lead to a variety of possible low dimensional theories depending on the relative size of the metric changes imposed on the system. This potential multiplicity of asymptotic theories is of course presaged by a similar state of affairs for the derivation of a nonlinear theory of elastic shells [8, 18].

Following the study of the *nonlinear elasticity* for thin plates and shells using a rigorous asymptotic theory [7, 6, 16, 17, 18] and a linearized theory [19] for residually strained Kirchhoff plates [12], we recently provided a rigorous derivation for a nonlinear Föppl-von Kármán like theory for the growth of naturally flat plates [15]. The resulting theory was consistent with that postulated in [20] using a thermoelastic analogy, and derived formally from 3d elasticity using perturbation theory [4, 5]. Here, we continue and extend the discussion in [20, 21, 15] concerning the rigorous derivation of asymptotic theories for the shape of *residually strained* thin lamina with nontrivial curvatures.

As our starting point for a similar theory for growing curved shells, we use the observation that it is possible to change the shape of a lamina (a blooming lily petal) by driving it via excess growth of the margins relative to the center, rather than via the midrib deformation, as conjectured in previous studies [28]. Again, a thermoelastic analogy [22] suggests a natural generalization of the Donnell-Mushtari-Vlasov shell theory [1] to growing shells [21] proposed as a mathematical model for blooming, activated from the initial (transverse) out-of-plane displacement v_0 of a petal's mid-surface, with respect to a flat reference configuration. When $v_0 = 0$ the equations (3.1) reduce to the prestrained von Kármán equations (2.7) proposed in [20] and rigorously derived in [15] from the so-called *non-Euclidean elasticity*, where the imposed 3d prestrain is given via a Riemannian metric, whose components display the appropriate target (linear) stretching tensor ϵ_g (of order 2 in shell's thickness h), and the bending tensor κ_g (of order 1 in h , see (2.1)).

In this paper we perform a rigorous analysis of the 3d non-Euclidean elasticity model and derive reduced-order theories for elastic shells that are weakly strained by differential growth. In Section 2 we show that for non-flat mid-surface (of arbitrary curvature and when $v_0 \neq 0$ is arbitrarily large), the variationally correct limiting theory (2.3) is an extension of the classical *von Kármán energy* to shells in [16]. In the special case $v_0 = 0$, this energy still reduces to the functional whose Euler-Lagrange equations give precisely (2.7).

In Section 3, we show that the equations (3.1) can be formally derived from (2.7) by pulling back the prestrain tensors ϵ_g and κ_g from the shallow shells whose reference mid-surface is given by the out-of-plane displacement hv_0 , hence comparable with the shell's thickness h . We then show (see Theorem 3.3) that (3.1) is the Euler-Lagrange equations of the variational limit for the 3d nonlinear elastic energies on weakly prestrained *shallow shells*, where the order of the prestrain tensor is appropriately compatible with the order of the mid-surface curvature.

In Section 4, we consider a more general situation, where the referential mid-surface is given by the out-of-plane displacement $h^\alpha v_0$, for an arbitrary exponent $\alpha > 0$. When $\alpha < 1$,

we prove in Theorem 4.1 that in the limit the shallowness takes over the prestrain, i.e. there is no effect of the tensors ϵ_g, κ_g on the limiting variational lower bound of the energy. This energy, given in (4.3) consists of the linearized Kirchhoff-type [8] functional which minimizes the relative bending under the constraint (4.4) that the limiting displacement v and the original profile v_0 have the same determinants of their Hessians. This amounts to saying that the image surfaces of $id + hv_0e_3$ and $id + hve_3$ have the same Gaussian curvatures, up to the highest order terms.

We conjecture that the same energy is also the variational upper bound, when $\alpha < 1$. As a first step towards establishing this natural claim, and in order to shed light on the mentioned admissibility constraint, we prove Theorem 4.3 corresponding to a *matching property* which states that v is admissible if and only if it can be matched by equibounded higher order displacements to be *isometrically equivalent* with v_0 . For results similar in spirit but in different contexts see [8, 27, 17, 10]. The two essential assumptions of Theorem 4.3 are: regularity $v \in \mathcal{C}^{2,\beta}$, and convexity $\det \nabla^2 v_0 > 0$.

We omit most of the proofs in the paper, since they consist of tedious yet only minor from the analytical viewpoint, necessary modifications of the arguments in [16, 15]. The result in Theorem 4.3 is however completely new, hence we present its proof entirely, in Section 5.

2. THE PRESTRAINED VON KÁRMÁN ENERGY ON SHELLS OF ARBITRARY CURVATURE

Let S be a 2-dimensional surface embedded in \mathbb{R}^3 . We assume that S is compact, connected, oriented, and of class $\mathcal{C}^{1,1}$, and that its boundary ∂S is the union of finitely many (possibly none) Lipschitz continuous curves. Consider a family $\{S^h\}_{h>0}$ of thin shells of thickness h around S :

$$S^h = \left\{ z = x + t\vec{n}(x); x \in S, -\frac{h}{2} < t < \frac{h}{2} \right\}, \quad 0 < h < h_0 \ll 1$$

where we use the following notation: $\vec{n}(x)$ for the unit normal, $T_x S$ for the tangent space, and $\Pi(x) = \nabla \vec{n}(x)$ for the shape operator on S , at a given $x \in S$. The projection onto S along \vec{n} is denoted by π , so that $\pi(z) = x$ for all $z = x + t\vec{n}(x) \in S^h$, and we assume that $h \ll 1$ is small enough to have π well defined on each S^h .

The instantaneous growth of S^h is described by two smooth tensors: $\epsilon_g, \kappa_g : \bar{S} \rightarrow \mathbb{R}^{3 \times 3}$:

$$(2.1) \quad a^h = [a_{ij}^h] : \bar{S}^h \rightarrow \mathbb{R}^{3 \times 3}, \quad a^h(x + t\vec{n}) = \text{Id} + h^2 \epsilon_g(x) + ht \kappa_g(x).$$

We note that the growth tensor a^h is as in [20, 15], but now in a general non-flat geometry setting.

For an elastic body, the energy density $W : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}_+$, in addition to being \mathcal{C}^2 regular in a neighborhood of $SO(3)$, is only assumed to satisfy normalization, frame indifference and nondegeneracy conditions:

$$\begin{aligned} \exists c > 0 \quad \forall F \in \mathbb{R}^{3 \times 3} \quad \forall R \in SO(3) \quad W(R) = 0, \quad W(RF) = W(F), \\ W(F) \geq c \, \text{dist}^2(F, SO(3)). \end{aligned}$$

where $F = \nabla u$ is the deformation gradient associated with a mapping u . Consequently, the thickness averaged elastic energy of a deformation u^h is given by:

$$(2.2) \quad I^h(u^h) = \frac{1}{h} \int_{S^h} W(\nabla u^h(a^h)^{-1}) \, dz, \quad \forall u \in W^{1,2}(S^h, \mathbb{R}^3).$$

Introducing the asymptotic limit (as $h \rightarrow 0$) of the scaled energies I^h then leads, following Theorems 2.1 and 2.2, to the variationally correct model for weakly prestrained shells with arbitrary large curvature, and corresponds to a nonlinear energy functional \mathcal{I}_4 acting on admissible limiting pairs $(V \in \mathcal{V}, B \in \mathcal{B})$:

$$(2.3) \quad \begin{aligned} \forall V \in \mathcal{V} \quad \forall B \in \mathcal{B} \quad \mathcal{I}_4(V, B) = & \frac{1}{2} \int_S \mathcal{Q}_2 \left(x, B - \frac{1}{2}(A^2)_{tan} - (\text{sym } \epsilon_g)_{tan} \right) \\ & + \frac{1}{24} \int_S \mathcal{Q}_2 \left(x, (\nabla(A\vec{n}) - A\Pi)_{tan} - (\text{sym } \kappa_g)_{tan} \right). \end{aligned}$$

Here, the space \mathcal{V} consists of first-order infinitesimal isometries on S defined by:

$$(2.4) \quad \mathcal{V} = \left\{ V \in W^{2,2}(S, \mathbb{R}^3); \tau \cdot \partial_\tau V(x) = 0 \quad \forall \text{a.e. } x \in S \quad \forall \tau \in T_x S \right\},$$

that is those $W^{2,2}$ regular displacements for whom the change of metric on S due to the deformation $\text{id} + \epsilon V$ is of order ϵ^2 , as $\epsilon \rightarrow 0$. Furthermore, for a matrix field $A \in L^2(S, \mathbb{R}^{3 \times 3})$, by $A_{tan}(x)$ we denote the tangential minor of A at $x \in S$, that is $[(A(x)\tau)\eta]_{\tau, \eta \in T_x S}$. We note that the skew-symmetric gradient of V as in (2.4) uniquely determines a $W^{1,2}$ matrix field $A : S \rightarrow so(3)$ so that: $\partial_\tau V(x) = A(x)\tau$ for all $\tau \in T_x S$. Hence, we may alternatively write:

$$\begin{aligned} \mathcal{V} = \left\{ V \in W^{2,2}(S, \mathbb{R}^3); \quad \exists A \in W^{1,2}(S, \mathbb{R}^{3 \times 3}) \quad \forall \text{a.e. } x \in S \quad \forall \tau \in T_x S \right. \\ \left. \partial_\tau V(x) = A(x)\tau \text{ and } A(x)^T = -A(x) \right\}. \end{aligned}$$

For a plate, that is when $S \subset \mathbb{R}^2$, an equivalent analytic characterization for $V = (V^1, V^2, V^3) \in \mathcal{V}$ is given by: $(V^1, V^2) = (-\omega y, \omega x) + (b_1, b_2)$, while the out-of-plane displacement $V^3 \in W^{2,2}(S, \mathbb{R})$ remains unconstrained.

Moreover, the space \mathcal{B} in (2.3) consists of finite strains:

$$\mathcal{B} = \left\{ L^2 - \lim_{\epsilon \rightarrow 0} \text{sym} \nabla w^\epsilon; \quad w^\epsilon \in W^{1,2}(S, \mathbb{R}^3) \right\},$$

which are the limits of symmetrized gradients of all sequences of displacements on S . Above, by $\text{sym} \nabla w(x)$ we mean a bilinear form on $T_x S$ given by: $(\text{sym} \nabla w(x)\tau)\eta = \frac{1}{2}[(\partial_\tau w(x))\eta + (\partial_\eta w(x))\tau]$, for all $\tau, \eta \in T_x S$.

It follows (via Korn's inequality) that for a naturally flat plate $\mathcal{B} = \{\text{sym} \nabla w; w \in W^{1,2}(S, \mathbb{R}^2)\}$, i.e. \mathcal{B} consists precisely of symmetrized gradients of all the in-plane displacements of S . When S is strictly convex, rotationally symmetric, or developable without flat regions, it has been proven in [16] that $\mathcal{B} = L^2(S, \mathbb{R}_{sym}^{2 \times 2})$, i.e. it contains all symmetric matrix fields on S with square integrable entries.

Finally, in (2.3), the quadratic forms :

$$\mathcal{Q}_3(F) = D^2 W(\text{Id})(F, F), \quad \mathcal{Q}_2(x, F_{tan}) = \min\{\mathcal{Q}_3(\tilde{F}); \tilde{F} \in \mathbb{R}^{3 \times 3}, (\tilde{F} - F)_{tan} = 0\}.$$

where the form \mathcal{Q}_3 is defined for $F \in \mathbb{R}^{3 \times 3}$, while $\mathcal{Q}_2(x, \cdot)$, for a given $x \in S$ is defined on tangential minors F_{tan} of such matrices. Both forms \mathcal{Q}_3 and all $\mathcal{Q}_2(x, \cdot)$ are positive definite and depend only on the symmetric parts of their arguments.

We now have the following results, stating in particular that the functional \mathcal{I}_4 is the Γ -limit [3] of the scaled energies $h^{-4}I^h$:

Theorem 2.1. *Let a sequence of deformations $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ satisfy $I^h(u^h) \leq Ch^4$. Then there exists proper rotations $\bar{R}^h \in SO(3)$ and translations $c^h \in \mathbb{R}^3$ such that for the renormalized deformations:*

$$y^h(x + t\vec{n}(x)) = (\bar{R}^h)^T u^h(x + t\frac{h}{h_0}\vec{n}) - c^h : S^{h_0} \longrightarrow \mathbb{R}^3$$

defined on the common thin shell S^{h_0} , the following holds.

- (i) y^h converge in $W^{1,2}(S^{h_0}, \mathbb{R}^3)$ to π .
- (ii) The scaled displacements:

$$(2.5) \quad V^h(x) = h^{-1} \int_{-h_0/2}^{h_0/2} y^h(x + t\vec{n}) - x \, dt$$

converge (up to a subsequence) in $W^{1,2}(S, \mathbb{R}^3)$ to some $V \in \mathcal{V}$.

- (iii) The scaled averaged strains:

$$(2.6) \quad B^h(x) = h^{-1} \text{sym} \nabla V^h(x)$$

converge (up to a subsequence) weakly in $L^2(S, \mathbb{R}^{2 \times 2})$ to a limit $B \in \mathcal{B}$.

- (iv) The lower bound holds:

$$\liminf_{h \rightarrow 0} h^{-4} I^h(u^h) \geq \mathcal{I}_4(V, B).$$

Theorem 2.2. *For every couple $V \in \mathcal{V}$ and $B \in \mathcal{B}$, there exists a sequence of deformations $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ such that the following holds:*

- (i) The rescaled sequence $y^h(x + t\vec{n}) = u^h(x + t\frac{h}{h_0}\vec{n})$ converges in $W^{1,2}(S^{h_0}, \mathbb{R}^3)$ to π .
- (ii) The displacements V^h as in (2.5) converge in $W^{1,2}(S, \mathbb{R}^3)$ to V .
- (iii) The strains B^h as in (2.6) converge in $W^{1,2}(S, \mathbb{R}^{2 \times 2})$ to B .
- (iv) One has:

$$\lim_{h \rightarrow 0} h^{-4} I^h(u^h) = \mathcal{I}_4(V, B).$$

The proofs follow through a combination of the arguments in [15] and [16], and we leave them to the interested reader. We now comment on the functional (2.3) and its relation with the prestrained von Kármán equations for plates.

Here, in analogy with the theory for naturally flat plates with incompatible strains [15], in (2.1), we have assumed that the target metric is 2nd order in thickness for the in-plane stretching: $(\text{sym } \epsilon_g)$, and 1st order in the thickness for bending: $(\text{sym } \kappa_g)$. Due to this particular choice of scalings the limit energy \mathcal{I}_4 is composed of exactly two terms, corresponding to stretching and bending. The argument of the integrand in the first term: $B - \frac{1}{2}(A^2)_{tan} - (\text{sym } \epsilon_g)_{tan}$ is the difference of the second order stretching induced by the deformation $\text{id} + hV + h^2w^h$ from the target stretching $(\text{sym } \epsilon_g)$, where $V \in \mathcal{V}$ and

$\text{sym} \nabla w^h \rightarrow B$. The argument of the integrand in the second term: $(\nabla(A\vec{n}) - A\Pi)_{tan} - (\text{sym } \kappa_g)_{tan}$ is the difference of the first order bending induced by the same deformation from the target bending $(\text{sym } \kappa_g)$.

In general, the second order displacement w can be very oscillatory. Due to the non-trivial geometry of the mid-surface, the finite strain space \mathcal{B} is usually large. In other words, a bound on the L^2 norm of the symmetric gradients $\text{sym} \nabla w^h$ implies only a very weak bound on w^h , which means that the limiting B can be written only as the symmetric gradient of a very weakly regular distribution (not a classical higher order displacement).

Remark 2.3. In the case of a plate, i.e. when $S \subset \mathbb{R}^2$, the term $B - \frac{1}{2}(A^2)_{tan}$ is reduced to the classical case: $\frac{1}{2}(\nabla w + (\nabla w)^T + \nabla v \otimes \nabla v)$, where w and $v = V^3$ are respectively the in-plane and the out-of-plane displacements. The term $(\nabla(A\vec{n}) - A\Pi)_{tan}$ reduces also to: $-\nabla^2 v$. As shown in [15], under the assumption of W being isotropic, the Euler-Lagrange equations of \mathcal{I}_4 can be then written in terms of the displacement v and the Airy stress potential Φ :

$$(2.7) \quad \begin{cases} \Delta^2 \Phi = -S(\det \nabla^2 v + \lambda_g) \\ B \Delta^2 v = [v, \Phi] - B \Omega_g, \end{cases}$$

where S is the Young modulus, B the bending stiffness, ν the Poisson ratio (given in terms of the Lamé constants μ and λ), and :

$$(2.8) \quad \begin{aligned} \lambda_g &= \text{curl}^T \text{curl} (\epsilon_g)_{2 \times 2} = \partial_{22}(\epsilon_g)_{11} + \partial_{11}(\epsilon_g)_{22} - \partial_{12}((\epsilon_g)_{12} + (\epsilon_g)_{21}), \\ \Omega_g &= \text{div}^T \text{div} \left((\kappa_g)_{2 \times 2} + \nu \text{ cof } (\kappa_g)_{2 \times 2} \right) \\ &= \partial_{11}((\kappa_g)_{11} + \nu(\kappa_g)_{22}) + \partial_{22}((\kappa_g)_{22} + \nu(\kappa_g)_{11}) + (1 - \nu)\partial_{12}((\kappa_g)_{12} + (\kappa_g)_{21}). \end{aligned}$$

Equations (2.7) are based on a thermoelastic analogy to growth [22, 20] and also derived using a formal perturbation theory [4].

Remark 2.4. When the mid-surface S is elliptic, for any first order isometry $V \in \mathcal{V}$ one can take $B \in \mathcal{B} = L^2(S, \mathbb{R}_{sym}^{2 \times 2})$ such that $B - \frac{1}{2}(A^2)_{tan} - (\text{sym } \epsilon_g)_{tan} = 0$. This implies that for any first order isometry, there exists a higher order modification w^h for which in the limit, the second order target stretching is realized. Thus, the energy reduces to:

$$\mathcal{I}_4(V) = \frac{1}{24} \int_S \mathcal{Q}_2 \left(x, (\nabla(A\vec{n}) - A\Pi)_{tan} - (\text{sym } \kappa_g)_{tan} \right) dx,$$

i.e. the bending term to be minimized over the space \mathcal{V} . Note that this problem is convex (minimizing a convex integral over a linear space \mathcal{V}), and hence it admits only one solution (up to rigid motions). Following the analysis in [17], we see that for elliptic surfaces, all limiting theories for $h^{-\beta} I^h$ under the energy scaling $\beta > 2$, coincide with the linear theory \mathcal{I}_4 as above, while the sublinear theory, to be used for studying e.g. buckling, is the Kirchhoff-like (nonlinear bending) theory corresponding to $\beta = 2$ and described in [19].

3. THE PRESTRAINED SHALLOW SHELL MODEL.

For the case when the mid-surface of the slender structure is a shell with non-zero curvature, the following system was introduced in [21], again using a thermoelastic analogy to

growth, as a mathematical model of blooming activated by differential lateral growth from an initial non-zero transverse displacement field v_0 :

$$(3.1) \quad \begin{cases} \Delta^2 \Phi = -S(\det \nabla^2 v - \det \nabla^2 v_0 + \lambda_g) \\ B(\Delta^2 v - \Delta^2 v_0) = [v, \Phi] - B\Omega_g. \end{cases}$$

Above, the notation and the growth sources λ_g and Ω_g are as in (2.8). We will now show that (3.1) can be derived from the equations (2.7).

For a given out-of-plate displacement $v_0 \in W^{2,2}(\Omega, \mathbb{R}) \cap \mathcal{C}^1(\bar{\Omega})$ on an open, bounded subset $\Omega \subset \mathbb{R}^2$, consider a sequence of surfaces:

$$S_h = \phi_h(\Omega), \text{ where } \phi_h(x) = (x, hv_0(x)) \quad \forall x = (x_1, x_2) \in \Omega.$$

The unit normal vector to S_h at $\phi_h(x)$ is given by:

$$\vec{n}^h(x) = \frac{\partial_1 \phi_h(x) \times \partial_2 \phi_h(x)}{|\partial_1 \phi_h(x) \times \partial_2 \phi_h(x)|} = \frac{1}{\sqrt{1 + h^2 |\nabla v_0|^2}} (-h\partial_1 v_0(x), -h\partial_2 v_0(x), 1) \quad \forall x \in \Omega.$$

We now consider thin plates $\Omega^h = \Omega \times (-h/2, h/2)$ and thin shallow shells $(S_h)^h$:

$$(3.2) \quad (S_h)^h = \{\tilde{\phi}_h(x, x_3); x \in \Omega, x_3 \in (-h/2, h/2)\},$$

where the Kirchhoff-Love extension of ϕ_h on Ω^h is given by the formula:

$$(3.3) \quad \tilde{\phi}_h(x, x_3) = \phi_h(x) + x_3 \vec{n}^h(x) \quad \forall (x, x_3) \in \Omega^h.$$

Let now $\epsilon_g, \kappa_g : \bar{\Omega} \rightarrow \mathbb{R}_{sym}^{3 \times 3}$ be two given smooth, symmetric tensors. Then,

Proposition 3.1. *The system (3.1) can be derived from the equations (2.7) by pulling back the prestrain tensors ϵ_g and κ_g from a sequence of shallow shells $(S_h)^h$ generated by the vanishing out-of-plane displacements hv_0 .*

For each small h , define the growth tensors on $(S_h)^h$ by:

$$(3.4) \quad q^h(\phi_h(x) + x_3 \vec{n}^h(x)) = \text{Id} + h^2 \epsilon_g(x) + hx_3 \kappa_g(x) \quad \forall (x, x_3) \in \Omega^h.$$

The corresponding metric $p^h = (q^h)^2$ on $(S_h)^h$ then satisfies:

$$p^h(\phi_h(x) + x_3 \vec{n}^h(x)) = \text{Id} + 2h^2 \epsilon_g(x) + 2hx_3 \kappa_g(x) + \mathcal{O}(h^3).$$

Lemma 3.2. *The pull-back of the metric p^h through $\tilde{\phi}_h$ satisfies:*

$$\begin{aligned} \forall (x, x_3) \in \Omega^h \quad g^h(x, x_3) &= (\nabla \tilde{\phi}_h)^T (p^h \circ \tilde{\phi}_h) (\nabla \tilde{\phi}_h) \\ &= \text{Id} + h^2 \left(2\epsilon_g(x) + (\nabla v_0(x) \otimes \nabla v_0(x))^* \right) + 2hx_3 \left(\kappa_g(x) - (\nabla^2 v_0(x))^* \right) + \mathcal{O}(h^3), \end{aligned}$$

where $F^* \in \mathbb{R}^{3 \times 3}$ denotes the matrix whose only non-zero entries are in its 2×2 minor given by $F \in \mathbb{R}^{2 \times 2}$.

Proof. By a direct calculation, we obtain:

$$\begin{aligned} \partial_1 \tilde{\phi}_h &= (1 - x_3 h \partial_{11}^2 v_0, -x_3 h \partial_{12}^2 v_0, h \partial_1 v_0) + \mathcal{O}(h^3), \\ \partial_2 \tilde{\phi}_h &= (-x_3 h \partial_{12}^2 v_0, 1 - x_3 h \partial_{22}^2 v_0, h \partial_2 v_0) + \mathcal{O}(h^3), \\ \partial_3 \tilde{\phi}_h &= \vec{n}^h = (-h \partial_1 v_0, -h \partial_2 v_0, 1 - \frac{1}{2} h^2 |\nabla v_0|^2) + \mathcal{O}(h^3). \end{aligned}$$

Hence:

$$\begin{aligned} (\nabla \tilde{\phi}_h)^T (\nabla \tilde{\phi}_h) &= \text{Id} - 2x_3 h (\nabla^2 v_0)^* + h^2 (\nabla v_0 \otimes \nabla v_0)^* + \mathcal{O}(h^3) \\ (\nabla \tilde{\phi}_h)^T (2h^2 \epsilon_g + 2hx_3 \kappa_g) (\nabla \tilde{\phi}_h) &= 2h^2 \epsilon_g + 2hx_3 \kappa_g + \mathcal{O}(h^3), \end{aligned}$$

in view of $\nabla \tilde{\phi}_h = \text{Id} + \mathcal{O}(h)$, and the result follows. \blacksquare

Proof of Proposition 3.1.

By Lemma 3.2 we see that the growth tensor on Ω^h has the form:

$$(3.5) \quad a^h = \sqrt{g^h} = \text{Id} + h^2 \left(\epsilon_g + \frac{1}{2} (\nabla v_0 \otimes \nabla v_0)^* \right) + hx_3 \left(\kappa_g - (\nabla^2 v_0)^* \right) + \mathcal{O}(h^3).$$

Applying (2.8) to the modified strain and curvature in a^h , to the leading order, we obtain:

$$\begin{aligned} \lambda_g(v_0) &= \text{curl}^T \text{curl} \left((\epsilon_g)_{2 \times 2} + \frac{1}{2} \nabla v_0 \otimes \nabla v_0 \right) = \lambda_g + \det \nabla^2 v_0 \\ \Omega_g(v_0) &= \text{div}^T \text{div} \left((\kappa_g - \nabla^2 v_0)_{2 \times 2} + \nu \text{ cof } (\kappa_g - \nabla^2 v_0)_{2 \times 2} \right) \\ &= \Omega_g - \Delta^2 v_0, \end{aligned}$$

where the last equality follows from $\text{div cof } \nabla^2 v_0 = 0$. Consequently, (2.7) for the growth tensor (3.5) becomes exactly (3.1). \blacksquare

We will now show that (3.1) can be rigorously derived as the Euler-Lagrange equations for the Γ -limit functional of the scaled 3d nonlinear elastic energies on weakly prestrained shallow shells. Let $v^h = u^h \circ \tilde{\phi}_h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$, by means of diffeomorphisms $\tilde{\phi}_h$ as in (3.3). By a simple change of variables, we see that:

$$\begin{aligned} I^h(u^h) &= \frac{1}{h} \int_{\Omega^h} W \left((\nabla v^h) (\nabla \tilde{\phi}_h)^{-1} (q^h \circ \tilde{\phi}_h)^{-1} \right) \cdot \det \nabla \tilde{\phi}_h \, dx \\ &= \frac{1}{h} \int_{\Omega^h} W \left((\nabla v^h) (b^h)^{-1} \right) \cdot \det \nabla \tilde{\phi}_h \, dx, \end{aligned}$$

where $b^h = (q^h \circ \phi_h) \nabla \tilde{\phi}_h$ satisfies

$$(b^h)^T b^h = g^h$$

and therefore by the polar decomposition of matrices

$$b^h = R(x, x_3) a^h \quad \text{on } \Omega^h$$

for some $R(x, x_3) \in SO(3)$ and the symmetric growth tensor a^h is given by (3.5).

For W isotropic, it follows that:

$$\begin{aligned} (3.6) \quad I^h(u^h) &= \frac{1}{h} \int_{\Omega^h} W \left((\nabla v^h) (a^h)^{-1} R(x)^{-1} \right) \cdot \det \nabla \tilde{\phi}_h \, dx \\ &= \frac{1}{h} \int_{\Omega^h} W \left((\nabla v^h) (a^h)^{-1} \right) \cdot (1 + \mathcal{O}(h)) \, dx. \end{aligned}$$

In other words, heuristically, and modulo the change of variable $\tilde{\phi}_h$, the problem reduces to the study of deformations of the flat domain Ω^h with the prestrain tensor a^h . Indeed, by exactly the same analysis as in [15], Theorems 1.2 and 1.3, we obtain:

Theorem 3.3. *Assume that $u^h \in W^{1,2}((S_h)^h, \mathbb{R}^3)$ satisfies $I^h(u^h) \leq Ch^4$. Then there exists proper rotations $\bar{R}^h \in SO(3)$ and translations $c^h \in \mathbb{R}^3$ such that for the normalized deformations:*

$$(3.7) \quad y^h(x, t) = (\bar{R}^h)^T(u^h \circ \tilde{\phi}_h)(x, ht) - c^h : \Omega^1 \longrightarrow \mathbb{R}^3$$

defined by means of (3.3) on the common domain $\Omega^1 = \Omega \times (-1/2, 1/2)$ the following holds:

- (i) $y^h(x, t)$ converge in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ to x .
- (ii) the scaled displacements $V^h(x) = h^{-1} \int_{-1/2}^{1/2} y^h(x, t) - x \, dt$ converge (up to a subsequence) in $W^{1,2}(\Omega, \mathbb{R}^3)$ to the vector field of the form $(0, 0, v)^T$ and $v \in W^{2,2}(\Omega, \mathbb{R})$.
- (iii) The scaled in-plane displacements $h^{-1} V_{tan}^h$ converge (up to a subsequence) weakly in $W^{1,2}$ to $w \in W^{1,2}(\Omega, \mathbb{R}^2)$.
- (iv) Moreover: $\liminf_{h \rightarrow 0} h^{-4} I^h(u^h) \geq \mathcal{I}_{g,v_0}(w, v)$ where:

$$(3.8) \quad \begin{aligned} \mathcal{I}_{g,v_0}(w, v) = & \frac{1}{2} \int_{\Omega} \mathcal{Q}_2 \left(\text{sym} \nabla w + \frac{1}{2} \nabla v \otimes \nabla v - \frac{1}{2} \nabla v_0 \otimes \nabla v_0 - (\epsilon_g)_{2 \times 2} \right) \\ & + \frac{1}{24} \int_{\Omega} \mathcal{Q}_2 \left(\nabla^2 v - \nabla^2 v_0 + (\kappa_g)_{2 \times 2} \right). \end{aligned}$$

The Euler-Lagrange equations of \mathcal{I}_{g,v_0} are precisely (3.1).

In the same manner, applying the proof of Theorem 1.4 of [15] to (3.6), yields:

Theorem 3.4. *For every $v \in W^{2,2}(\Omega, \mathbb{R})$ and $w \in W^{1,2}(\Omega, \mathbb{R}^2)$, there exists a sequence of deformations $u^h \in W^{1,2}((S_h)^h, \mathbb{R}^3)$ such that:*

- (i) The sequence $y^h(x, t) = u^h(x + hv_0(x)e_3 + ht\bar{n}^h(x))$ converges in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ to x .
- (ii) The displacements V^h as in (ii) Theorem 3.8 converge in $W^{1,2}$ to $(0, 0, v)$.
- (iii) The in-plane displacements $h^{-1} V_{tan}^h$ converge in $W^{1,2}$ to w .
- (iv) $\lim_{h \rightarrow 0} h^{-4} I^h(u^h) = \mathcal{I}_{g,v_0}(w, v)$.

Remark 3.5. Comparing functional (3.8) with (2.3), note that the space $\mathcal{V}_1(S_h)$ of first-order infinitesimal isometries on S_h is made of displacements $V : S_h \rightarrow \mathbb{R}^3$ of the form:

$$(3.9) \quad \begin{aligned} V(\phi_h(x)) &= (-hv_1(x), -hv_2(x), v_3) \quad \forall x \in \Omega, \\ \text{such that } (v_1, v_2, v_3) &\in W^{2,2}(\Omega, \mathbb{R}^3) \text{ and } \text{sym} \nabla(v_1, v_2) = \text{sym}(\nabla v_3 \otimes \nabla v_0). \end{aligned}$$

Indeed, the condition $\text{sym} \nabla V = 0$ on S_h becomes:

$$0 = \frac{1}{2} (\partial_i(V \circ \phi_h) \cdot \partial_j \phi_h + \partial_j(V \circ \phi_h) \cdot \partial_i \phi_h) = -h \text{sym}[\nabla(v_1, v_2) - \nabla v_3 \otimes \nabla v_0]_{ij}.$$

We also see that v_3 can be completed by (v_1, v_2) to $V \in \mathcal{V}_1(S_h)$ as in (3.9) only if:

$$(3.10) \quad \text{cof} \nabla^2 v_0 : \nabla^2 v_3 = 0,$$

the latter being also a sufficient condition when S is simply connected. This follows from:

$$\begin{aligned} \text{curl}^T \text{curl} \left(\text{sym}(\nabla v_3 \otimes \nabla v_0) \right) &= \text{curl}^T \text{curl} \left(\nabla v_3 \otimes \nabla v_0 \right) \\ &= \partial_{22}(\partial_1 v_3 \cdot \partial_1 v_0) + \partial_{11}(\partial_2 v_3 \cdot \partial_2 v_0) - \partial_{12}(\partial_1 v_3 \cdot \partial_2 v_0 + \partial_2 v_3 \cdot \partial_1 v_0) \\ &= -(\partial_{11} v_3 \cdot \partial_{22} v_0 + \partial_{22} v_3 \cdot \partial_{11} v_0 - 2\partial_{12} v_3 \cdot \partial_{12} v_0) = -\text{cof} \nabla^2 v_0 : \nabla^2 v_3. \end{aligned}$$

Hence, we see that the admissible out-of-plane displacements v_3 relevant in (2.3), must obey for the least the constraint (3.10), in contrast with the unconstrained 2-scale limiting theory (3.8).

4. SHALLOW SHELLS OF LOWER ORDERS

More generally, for any exponent $\alpha > 0$ we may consider the following energies of deformations on weakly prestrained shallow shells:

$$(4.1) \quad I^h(u^h) = \frac{1}{h} \int_{(S_h)^h} W((\nabla u^h)(q^h)^{-1}) \quad \forall u^h \in W^{1,2}((S_h)^h, \mathbb{R}^3),$$

with the growth tensor q^h given by (3.4), on thin shells of the form (3.2) around the shallow mid-surface:

$$S_h = \phi_h(\Omega); \quad \phi_h(x) = (x, h^\alpha v_0(x)); \quad v_0 \in \mathcal{C}^{2,\beta}(\bar{\Omega}, \mathbb{R}).$$

According to our notation, $\vec{n}^h(x)$ stands for the unit normal vector to S_h at $\phi_h(x)$, i.e.: $\vec{n}^h(x) = (\partial_1 \phi_h(x) \times \partial_2 \phi_h(x)) / |\partial_1 \phi_h(x) \times \partial_2 \phi_h(x)|$.

When $\alpha > 1$, it is clear from the previous calculations that the prestrain takes over the effect of shallowness and hence the limiting theory is that derived in [15], coinciding with results of Theorem 3.3 for the case $v_0 = 0$ and yielding the Euler-Lagrange equations (2.7).

Assume now that $0 < \alpha < 1$. By a straightforward calculation, we see that $I^h(u^h)$ can be written as in (3.6), with:

$$a^h = \text{Id} + \frac{1}{2} h^{2\alpha} (\nabla v_0 \otimes \nabla v_0)^* - h^\alpha x_3 (\nabla^2 v_0)^* + o(h^{2\alpha}) + x_3 o(h^\alpha)$$

where the uniform quantities in $o(h^{2\alpha})$, $o(h^\alpha)$ are independent of x_3 . Following the proof of Theorem 1.3 in [15], we obtain:

Theorem 4.1. *Assume that $u^h \in W^{1,2}((S_h)^h, \mathbb{R}^3)$ satisfies $I^h(u^h) \leq Ch^{2\alpha+2}$, where I^h is given in (4.1) and $0 < \alpha < 1$. Then there exists $\bar{R}^h \in SO(3)$ and translations $c^h \in \mathbb{R}^3$ such that for y^h as in (3.7), we have:*

- (i) $y^h(x, t)$ converge in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ to x .
- (ii) the scaled displacements $V^h(x) = h^{-\alpha} \int_{-1/2}^{1/2} y^h(x, t) - x \, dt$ converge (up to a subsequence) in $W^{1,2}(\Omega, \mathbb{R}^3)$ to $(0, 0, v)^T$ where $v \in W^{2,2}(\Omega, \mathbb{R})$ and:

$$(4.2) \quad \det \nabla^2 v = \det \nabla^2 v_0.$$

- (iii) Moreover: $\liminf_{h \rightarrow 0} h^{-(2\alpha+2)} I^h(u^h) \geq \mathcal{I}_{v_0}(w, v)$ where:

$$(4.3) \quad \mathcal{I}_{v_0}(v) = \frac{1}{24} \int_{\Omega} \mathcal{Q}_2(\nabla^2 v - \nabla^2 v_0).$$

The constraint in the assertion (ii) follows by observing that $h^{-\alpha} \text{sym} \nabla V^h$ converges in $L^2(\Omega)$ to $\frac{1}{2} (\nabla v_0 \otimes \nabla v_0 - \nabla v \otimes \nabla v)$ which implies (4.2).

We now conjecture that the linearized Kirchhoff-like energy (4.3) (4.2) is also the exact Γ -limit of the rescaled energies $h^{-(2\alpha+2)} I^h(u^h)$ when Ω is simply connected. If Ω is not simply connected, one must replace (4.2) with a more general variant which states that $\nabla v_0 \otimes \nabla v_0 - \nabla v \otimes \nabla v$ is a symmetric gradient. As a first step towards the rigorous proof

of this statement, and to shed light on the constraint condition (4.2), we now derive a matching property for weakly shallow shells. Let us first explain the meaning of “matching property” and put our analysis in a broader context.

Remark 4.2. In [18], the authors put forward a conjecture regarding existence of infinitely many small slope shell theories (with no prestrain) each valid for a corresponding range of energy scalings. This conjecture is based on formal asymptotic expansions and it is in accordance with the previously obtained results for plates and shells. It predicts the form of the 2-dimensional limit energy functional, and identifies the space of admissible deformations as infinitesimal isometries of a given integer order $N > 0$ determined by the magnitude of the elastic energy. Hence, the influence of shells geometry on its qualitative response to an external force, i.e. the shells rigidity, is reflected in a hierarchy of functional spaces of isometries (and infinitesimal isometries) arising as constraints of the derived theories.

In certain cases, a given N th order infinitesimal isometry can be modified by higher order corrections to yield an infinitesimal isometry of order $M > N$, a property to which we refer to by “matching property of infinitesimal isometries”. This feature, combined with certain density results for spaces of isometries, cause the theories corresponding to orders of infinitesimal isometries between N and M , to collapse all into one and the same theory. The examples of such behavior are observed for plates [8], where any second order infinitesimal isometry can be matched to an exact isometry ($M = \infty$), for convex shells [17], where any first order infinitesimal isometry satisfies the same property, and for non-flat developable surfaces [27, 10] where first order isometries can be matched to higher order isometries. The effects of these geometric properties on the elasticity of thin films are drastic. A plate whose boundary is at least partially free possesses three types of small-slope theories: the linear theory, the von Kármán theory and the linearized Kirchhoff theory, whereas the only small slope theory for a convex shell with free boundary is the linear theory [17]: a convex shell transitions directly from the linear regime to the fully nonlinear bending one if the applied forces are adequately increased. In other words, while the von Kármán theory describes the buckling of thin plates at a body force magnitude of order thickness-cubed, the equivalent, variationally correct theory for buckling of elliptic shells is the purely nonlinear bending theory which comes only into effect when the body forces reach to a magnitude of order thickness-squared.

In the present context, we have the following:

Theorem 4.3. *Assume that Ω is simply connected and let $v_0 \in \mathcal{C}^{2,\beta}(\Omega, \mathbb{R})$ with $\det \nabla^2 v_0 > c > 0$. Let $v \in \mathcal{C}^{2,\beta}(\Omega, \mathbb{R})$ satisfy:*

$$(4.4) \quad \det \nabla^2 v = \det \nabla^2 v_0 \quad \text{in } \Omega.$$

Then there exists a sequence $w_h \in \mathcal{C}^{2,\beta}(\Omega, \mathbb{R}^3)$ such that:

$$(4.5) \quad \forall h > 0 \quad \nabla(\text{id} + hve_3 + h^2w_h)^T \nabla(\text{id} + hve_3 + h^2w_h) = \nabla(\text{id} + hv_0e_3)^T \nabla(\text{id} + hv_0e_3)$$

and $\sup \|w_h\|_{\mathcal{C}^{2,\beta}} < +\infty$.

Before giving the proof in Section 5, we present a brief discussion. Writing $w_h = w_{h,tan} + w_h^3 e_3$ where $w_{h,tan}(x) \in \mathbb{R}^2$ and $w_h^3 \in \mathbb{R}$, equation (4.5) becomes:

$$(4.6) \quad \begin{aligned} & \text{Id} + h^2(2\text{sym}\nabla w_{h,tan} + \nabla v \otimes \nabla v) + 2h^3\text{sym}(\nabla v \otimes \nabla w_h^3) \\ & + h^4((\nabla w_{h,tan})^T \nabla w_{h,tan} + \nabla w_h^3 \otimes \nabla w_h^3) = \text{Id} + h^2 \nabla v_0 \otimes \nabla v_0. \end{aligned}$$

Recall that (4.4) is equivalent to $\text{curl}^T \text{curl}(\nabla v \otimes \nabla v - \nabla v_0 \otimes \nabla v_0) = 0$, and hence to: $\nabla v \otimes \nabla v - \nabla v_0 \otimes \nabla v_0 = \text{sym}\nabla w$ for some $w \in \mathcal{C}^{2,\beta}(\Omega, \mathbb{R}^2)$ since Ω is simply connected. Hence the constraint (4.4) is necessary and sufficient for matching the lowest order (h^2) terms in (4.6).

Our result states that actually it is possible to perturb w by an equibounded 3d displacement $w_h - w$ so that the full equality (4.6) holds. A natural way for proving this is by implicit function theorem. Indeed, this is how we proceed, and the ellipticity assumption $\det \nabla^2 v_0 > 0$ is precisely a sufficient condition for the invertibility of the implicit derivative $\mathcal{L}(p) : \mathcal{C}_0^{2,\beta}(\Omega, \mathbb{R}) \rightarrow \mathcal{C}^{0,\beta}(\Omega, \mathbb{R})$, $\mathcal{L}(p) = -\text{cof}\nabla^2 v : \nabla^2 p$ where p is the variation in w_h^3 . An extra argument for the uniform boundedness of $w_{h,tan}$ in $\mathcal{C}^{2,\beta}$ concludes the proof.

Remark 4.4. The condition (4.5) means that each deformation $u^h : S_h \rightarrow \mathbb{R}^3$ of a surface $S_h = \{x + hv_0(x)e_3; x \in \Omega\}$, given by $u^h(x + hv_0(x)e_3) = x + hv(x)e_3 + h^2 w_h(x)$ is an isometry of S_h . In other words, the pull-back metrics from the Euclidean metric of S_h and of $u^h(S_h) = \{x + hv(x)e_3 + h^2 w_h(x); x \in \Omega\}$ coincide. Hence Theorem 4.3 asserts that if two convex out-of-plane displacements of first order have the same determinants of Hessians, then they can be matched by a family of equibounded higher order displacements (the fields w_h) to be isometrically equivalent. When the parameter h is replaced by h^α , this result can be also viewed in the context of shallow shells. For other results concerning matching of isometries see [8, Theorem 7], [17, Theorem 1.1] and [10, Theorem 3.1] (which is comparable with [27, Lemma 3.3] and the remark which follows therein).

5. A MATCHING PROPERTY ON CONVEX SHALLOW SHELLS: PROOF OF THEOREM 4.3.

By a direct calculation, (4.5) is equivalent to:

$$(5.1) \quad \nabla(\text{Id} + h^2 w_{h,tan})^T \nabla(\text{Id} + h^2 w_{h,tan}) = \text{Id} + h^2 \nabla v_0 \otimes \nabla v_0 - h^2(\nabla v + \nabla z_h) \otimes (\nabla v + \nabla z_h),$$

where $w_{h,tan} \in \mathcal{C}^{2,\beta}(\Omega, \mathbb{R}^2)$ and $z_h = hw_h^3 \in \mathcal{C}^{2,\beta}(\Omega, \mathbb{R})$ so that $w_h = w_{h,tan} + w_h^3 e_3$ is the required correction in (4.5).

1. We shall first find the formula for the Gaussian curvature of the 2d metric in the right hand side of (5.1):

$$(5.2) \quad g_h(z_h) = \text{Id} + h^2 \nabla v_0 \otimes \nabla v_0 - h^2(\nabla v + \nabla z_h) \otimes (\nabla v + \nabla z_h).$$

Lemma 5.1. *Let $v_0, v \in \mathcal{C}^{2,\beta}(\Omega, \mathbb{R})$ and consider the $\mathcal{C}^{1,\beta}$ regular metrics on Ω of the type:*

$$g = [g_{ij}]_{i,j=1,2} = \text{Id} + h^2(\nabla v_0 \otimes \nabla v_0 - \nabla v_1 \otimes \nabla v_1).$$

Then, for any $h > 0$ small, the Gaussian curvature $\kappa(g)$ of g is $\mathcal{C}^{0,\beta}$ regular and it is given by the formula:

$$(5.3) \quad \kappa(g) = h^2 \left[\frac{\det(\nabla^2 v_0 - [\Gamma_{ij}^k \partial_k v_0]_{ij})}{(1 - h^2(g^{ij} \partial_i v_0 \partial_j v_0))^2 \det g} - \frac{(1 - h^2(g^{ij} \partial_i v_0 \partial_j v_0))^2}{(1 - h^2|\nabla v_1|^2)^2} \det \nabla^2 v_1 \right],$$

where the Christoffel symbols of g , the inverse of g , and its determinant are:

$$(5.4) \quad \Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_j g_{il} + \partial_i g_{jl} - \partial_j g_{ij}),$$

$$(5.5) \quad g^{-1} = [g^{ij}] = \frac{1}{\det[g_{ij}]} \text{cof}[g_{ij}],$$

$$\det g = 1 - h^4 |(\nabla v_0)^\perp \cdot \nabla v_1|^2 + h^2 (|\nabla v_0|^2 - |\nabla v_1|^2).$$

Proof. Assume first that v_0 and v_1 are in fact smooth. By Lemma 2.1.2 in [9], we have:

$$\begin{aligned} \kappa(\text{Id} - h^2 \nabla v_1 \otimes \nabla v_1) &= -h^2 \frac{\det \nabla^2 v_1}{(1 - h^2 |\nabla v_1|^2)^2} \\ \kappa(g - h^2 \nabla v_0 \otimes \nabla v_0) &= \frac{1}{(1 - h^2 (g^{ij} \partial_i v_0 \partial_j v_0))^2} \left[\kappa(g) - \frac{h^2 \det(\nabla^2 v_0 - [\Gamma_{ij}^k \partial_k v_0]_{ij})}{(1 - h^2 (g^{ij} \partial_i v_0 \partial_j v_0))^2 \det g} \right]. \end{aligned}$$

Since the two metrics above are equal, the formula (5.3) follows directly. The formula for $\det g$ is obtained by a direct calculation, via $\det(A + B) = \det A + \text{cof} A : B + \det B$, valid for 2×2 matrices A, B .

In the general case when v_0, v_1 are only $\mathcal{C}^{2,\beta}$ regular, one may approximate them by smooth sequences v_0^n, v_1^n . Then, each $\kappa_n = \kappa(\text{Id} + h^2(\nabla v_0^n \otimes v_0^n - \nabla v_1^n \otimes v_1^n))$ is given by the formula in (5.3), and the sequence κ_n converges in $\mathcal{C}^{0,\beta}$ to the right hand side in (5.3). On the other hand, κ_n converges in $\mathcal{D}'(\Omega)$ to $\kappa(g)$, which follows from the definition of Gauss curvature $\kappa = R_{1212}/\det g$. Hence the lemma is proven. \blacksquare

2. Applying Lemma 5.1 to $v_1 = v + z_h$, we now see that for small h , the Gauss curvature of metric $g_h(z_h)$ vanishes:

$$(5.6) \quad \kappa(g_h(z_h)) = 0$$

if and only if:

$$(5.7) \quad \Phi(h, z_h) = 0,$$

where:

$$\begin{aligned} \Phi(h, z) &= (1 - h^2 |\nabla v + \nabla z|^2)^2 \det(\nabla^2 v_0 - [\Gamma_{ij}^k \partial_k v_0]_{ij}) \\ &\quad - (1 - h^2 (g^{ij} \partial_i v_0 \partial_j v_0))^4 d(h, z) \det(\nabla^2 v + \nabla^2 z). \end{aligned}$$

Here:

$$d(h, z) = 1 - h^4 |(\nabla v_0)^\perp \cdot \nabla(v + z)|^2 + h^2 (|\nabla v_0|^2 - |\nabla v + \nabla z|^2)$$

and Γ_{ij}^k and g^{ij} are given by (5.4) and (5.5) for the metric $g = \text{Id} + h^2 \nabla v_0 \otimes \nabla v_0 - h^2 (\nabla v + \nabla z) \otimes (\nabla v + \nabla z)$. We shall consider:

$$\Phi : (-\epsilon, \epsilon) \times \mathcal{C}_0^{2,\beta}(\Omega, \mathbb{R}) \longrightarrow \mathcal{C}^{0,\beta}(\Omega, \mathbb{R})$$

and seek for solutions $z_h \in \mathcal{C}_0^{2,\beta}(\Omega, \mathbb{R})$ of (5.7) with zero boundary data. It is elementary to check that Φ is continuously Frechet differentiable at $(0, 0)$ and that

$$\Phi(0, 0) = \det \nabla^2 v_0 - \det \nabla^2 v = 0.$$

Moreover, the partial Frechet derivative $\mathcal{L} = \partial\Phi/\partial z(0, 0) : \mathcal{C}_0^{2,\beta}(\Omega, \mathbb{R}) \longrightarrow \mathcal{C}^{0,\beta}(\Omega, \mathbb{R})$ is a linear continuous operator of the form:

$$\begin{aligned} \forall z \in \mathcal{C}_0^{2,\beta} \quad \mathcal{L}(z) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \Phi(0, \epsilon z) = \lim_{\epsilon} \frac{1}{\epsilon} (\det \nabla^2 v_0 - \det(\nabla^2 v + \epsilon \nabla^2 z)) \\ &= \lim_{\epsilon} \frac{1}{\epsilon} (-\epsilon^2 \det \nabla^2 z - \epsilon \operatorname{cof} \nabla^2 v : \nabla^2 z) = -\operatorname{cof} \nabla^2 v : \nabla^2 z. \end{aligned}$$

Clearly, \mathcal{L} is invertible to a continuous linear operator, because of the uniform ellipticity of the matrix field $\nabla^2 v$ which follows from the convexity assumption of $\det \nabla^2 v = \det \nabla^2 v_0$ being strictly positive. Thus, invoking the implicit function theorem we obtain the solution operator:

$$\mathcal{Z} : (-\epsilon, \epsilon) \longrightarrow \mathcal{C}_0^{2,\beta}(\Omega, \mathbb{R})$$

such that $z_h = \mathcal{Z}(h)$ satisfies (5.7). Moreover, \mathcal{Z} is differentiable at $h = 0$ and:

$$\mathcal{Z}'(0) = \mathcal{L}^{-1} \circ \left(\frac{\partial \Phi}{\partial h}(0, 0) \right) = 0,$$

because:

$$\frac{\partial \Phi}{\partial h}(0, 0) = (\operatorname{cof} \nabla^2 v_0) : \left[\left(\frac{\partial}{\partial h} \Gamma_{ij}^k \right) \partial_k v_0 \right]_{ij} + \frac{\partial}{\partial h} \det[\Gamma_{ij}^k \partial_k v_0]_{ij} - \left(\frac{\partial}{\partial h} d(0, 0) \right) \det \nabla^2 v = 0.$$

Consequently:

$$(5.8) \quad \|w_h^3\|_{\mathcal{C}^{2,\beta}} = \frac{1}{h} \|z_h\|_{\mathcal{C}^{2,\beta}} \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

3. In conclusion, we have so far obtained a uniformly bounded sequence of $\mathcal{C}_0^{2,\beta}$ out-of-plane displacements $w_h^3 = z_h/h$ such that the Gauss curvature (5.6) of the metric $g_h(z_h)$ in the right hand side of (5.1) is 0. By the result in [23] it follows that for each small h there exists exactly one (up to fixed rotations) orientation preserving isometric immersion $\phi_h \in \mathcal{C}^2(\Omega, \mathbb{R}^2)$ of $g_h(z_h)$:

$$(5.9) \quad \nabla \phi_h^T \nabla \phi_h = g_h(z_h); \quad \det \nabla \phi_h > 0.$$

What remains to be proven is that, in fact, $\phi_h = \operatorname{id} + h^2 w_{h,tan}$ with some $w_{h,tan}$ uniformly bounded in $\mathcal{C}^{2,\beta}(\Omega, \mathbb{R}^2)$.

It is a well known calculation (see [2, 23]) that (5.9) implies (is actually equivalent to):

$$(5.10) \quad \nabla^2 \phi_h - [\Gamma_{ij}^k \partial_k \phi_h]_{ij} = 0,$$

where Γ_{ij}^k are the Christoffel symbols (5.4) of the metric $g = g_h(z_h)$ in (5.2). By (5.9) $\|\nabla \phi_h\|_{L^\infty} \leq C$, and by (5.10) $\|\nabla^2 \phi_h\|_{L^\infty} \leq C$, hence $\|\phi_h\|_{\mathcal{C}^{2,\beta}} \leq C$. But Γ_{ij}^k are uniformly bounded (with respect to small h) in $\mathcal{C}^{0,\beta}$ so by (5.10) $\|\nabla^2 \phi_h\|_{\mathcal{C}^{0,\beta}} \leq C$ and thus:

$$\|\phi_h\|_{\mathcal{C}^{2,\beta}(\Omega, \mathbb{R}^2)} \leq C.$$

Note now that $\|\Gamma_{ij}^k\|_{\mathcal{C}^{0,\beta}} \leq Ch^2$ in view of the particular structure of the metrics $g_h(z_h)$. Hence, by (5.10):

$$(5.11) \quad \|\nabla^2 \phi_h\|_{\mathcal{C}^{0,\beta}} \leq Ch^2.$$

Therefore, for some $A_h \in \mathbb{R}^{2 \times 2}$ we have:

$$(5.12) \quad \|\nabla \phi_h - A_h\|_{C^{1,\beta}} \leq Ch^2.$$

We now prove that the matrix A_h in the inequality above can be chosen as a rotation and hence, without loss of generality, $A_h = \text{Id}$. For each $x \in \Omega$ there holds:

$$(5.13) \quad \text{dist}(A_h, SO(3)) \leq |A_h - \nabla \phi_h(x)| + \text{dist}(\nabla \phi_h(x), SO(3)).$$

To evaluate the last term above, write: $\sqrt{\nabla \phi_h^T(x) \nabla \phi_h(x)} = QDQ^T$ for some $Q \in SO(3)$ and $D = \text{diag}(\lambda_1, \lambda_2)$ with $\lambda_1, \lambda_2 > 0$. Since $\det \nabla \phi_h > 0$, it follows by polar decomposition theorem that:

$$\begin{aligned} \text{dist}(\nabla \phi_h(x), SO(3)) &= |\sqrt{\nabla \phi_h^T(x) \nabla \phi_h(x)} - \text{Id}| \leq C|D - \text{Id}| \\ &= C \max_i \{|\lambda_i - 1|\} \leq C \max_i \{|\lambda_i^2 - 1|\} \leq C|D^2 - \text{Id}| \\ &= C|Q^T \nabla \phi_h^T(x) \nabla \phi_h(x) Q - \text{Id}| \leq C|\nabla \phi_h^T \nabla \phi_h(x) - \text{Id}| \leq Ch^2. \end{aligned}$$

By the above and (5.13), (5.12) we see that $\text{dist}(A_h, SO(3)) \leq Ch^2$. Hence, without loss of generality, $\|\nabla \phi_h - \text{Id}\|_{C^{1,\beta}} \leq Ch^2$ and:

$$\|\phi_h - \text{id}\|_{C^{2,\beta}} \leq Ch^2.$$

Consequently, $\phi_h = \text{id} + h^2 w_{h,tan}$ with $\|w_{h,tan}\|_{C^{2,\beta}} \leq C$. This concludes the proof of Theorem 4.3, in view of (5.1) which is equivalent to (4.5). \blacksquare

Remark 5.2. In view of the properties of the elliptic Monge-Ampère equation we expect that the $C^{2,\beta}$ scalar fields v_0 are dense in the set of the $W^{2,2}$ fields with a prescribed, strictly positive $\det \nabla^2$. With such a result at hand, it would indeed follow that for convex shells, the linearized Kirchhoff-type energy (4.3) is the rigorous variational limiting theory in the present context of weakly shallow shells, in the same spirit as the corresponding matching and density of first order isometries on convex shells [17] resulted in that the only small slope theory for an elastic convex shell is the linear theory. The latter problems, when posed for surfaces of arbitrary geometry, are more difficult. One could hope to be able to prove similar results in case S is a strictly hyperbolic surface. In its generality however they reduce to the study of nonlinear PDEs of mixed type for which not so many suitable methods are at hand.

6. DISCUSSION

Our analysis has shown how one can rigorously derive a general theory of shells with residual strain that might arise from relative growth, inhomogeneous swelling, plasticity etc. In fact, there are many such theories; each is a consequence of the scalings for the magnitude of the shallowness of the shell relative to the magnitude of the strain incompatibility induced by the metric and curvature growth tensors. Thus, not only do we recover a recently postulated model [21], but furthermore we also find a degenerate model in 4 that corresponds to a constrained shell theory with no contributions from growth. For simplicity of presentation, our analysis here is limited to a subset of vast possible scenarios since we fix the scalings of the incompatible strain tensors (2.1) or (3.4) throughout the paper. This situation might be contrasted with the case of the equations for the growth

of an elastic plate [15] where there seems to be just a single natural choice for the scalings which result in a generalized Föppl-von Kármán theory, but is similar to the situation for the weakly nonlinear theory of elastic shells where there are multiple potential scalings and corresponding theories.

A natural generalization of our results would be to allow for different scaling regimes for the growth tensors in search of other possible limiting theories. The same mathematical approach can be adapted to these other regimes with variants of Theorem 4.3 as an essential tool for a rigorous justification of other possible models. Overall, there are three independent parameters associated with scaling of the shallowness, and the two incompatible strains, characterized in terms of their dependence on the thickness in the form h^α ; the resulting theories depend on the choice of scalings for the three parameters. Thus, there is no single *correct* model in general, but of course when dealing with a concrete situation, a choice of particular scaling parameters for the relative magnitude of the thickness, the shallowness and the differential growth determine the effective theory.

Acknowledgments. M.L. is partially supported by the NSF grants DMS-0707275 and DMS-0846996, and by the Polish MN grant N N201 547438. L.M. is supported by the MacArthur Foundation, M.R.P. is partially supported by the NSF grants DMS-0907844 and DMS-1210258.

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